

# A MORE ACCURATE HALF-DISCRETE HARDY-HILBERT-TYPE INEQUALITY WITH THE BEST POSSIBLE CONSTANT FACTOR RELATED TO THE EXTENDED RIEMANN-ZETA FUNCTION

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## Abstract

By the method of weight coefficients, techniques of real analysis and Hermite-Hadamard's inequality, a half-discrete Hardy-Hilbert-type inequality related to the kernel of the hyperbolic cosecant function with the best possible constant factor expressed in terms of the extended Riemann-zeta function is proved. The more accurate equivalent forms, the operator expressions with the norm, the reverses and some particular cases are also considered.

**Key words:** Hardy-Hilbert-type inequality; extended Riemann-zeta function; Hurwitz zeta function; Gamma function; weight function; equivalent form; operator

## 1 Introduction

If  $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$ ,

$$\|f\|_p = \left( \int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} > 0,$$

and  $\|g\|_q > 0$ , then we have the following Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \quad (1.1)$$

where, the constant factor  $\frac{\pi}{\sin(\pi/p)}$  is the best possible. Assuming that

$$a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, \|a\|_p = \left( \sum_{m=1}^\infty a_m^p \right)^{\frac{1}{p}} > 0, \|b\|_q > 0,$$

we have the following discrete analogue of (1.1) with the same best constant  $\frac{\pi}{\sin(\pi/p)}$  (cf. [1]):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \quad (1.2)$$

Inequalities (1.1) and (1.2) are important in Mathematical Analysis and its applications (cf. [1], [2], [3], [4], [5]).

Suppose that  $\mu_i, \nu_j > 0$  ( $i, j \in \mathbf{N} = \{1, 2, \dots\}$ ),

$$U_m := \sum_{i=1}^m \mu_i, V_n := \sum_{j=1}^n \nu_j \quad (m, n \in \mathbf{N}). \quad (1.3)$$

Then we have the following inequality (cf. [1], Theorem 321, replacing  $\mu_m^{1/q} a_m$  and  $\nu_n^{1/p} b_n$  by  $a_m$  and  $b_n$ ):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left( \sum_{m=1}^\infty \frac{a_m^p}{\mu_m^{p-1}} \right)^{\frac{1}{p}} \left( \sum_{n=1}^\infty \frac{b_n^q}{\nu_n^{q-1}} \right)^{\frac{1}{q}}. \quad (1.4)$$

For  $\mu_i = \nu_j = 1$  ( $i, j \in \mathbf{N}$ ), inequality (1.4) reduces to (1.2). We call (1.4) Hardy-Hilbert-type inequality.

**Note.** The authors of [1] did not prove that (1.4) is valid with the best possible constant factor.

In 1998, by introducing an independent parameter  $\lambda \in (0, 1]$ , Yang [6] obtained an extension of (1.1) with the kernel  $\frac{1}{(x+y)^\lambda}$  for  $p = q = 2$ . Refining the method applied in [6], Yang [5] provided extensions of (1.1) and (1.2) as follows:

Assuming that  $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$  is a non-negative homogeneous function of degree  $-\lambda$ , with

$$\begin{aligned} k(\lambda_1) &= \int_0^\infty k_\lambda(t, 1) t^{\lambda_1-1} dt \in \mathbf{R}_+, \\ \phi(x) &= x^{p(1-\lambda_1)-1}, \quad \psi(x) = x^{q(1-\lambda_2)-1}, \quad f(x), g(y) \geq 0, \\ f &\in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; \|f\|_{p,\phi} := \left\{ \int_0^\infty \phi(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\}, \end{aligned}$$

where  $g \in L_{q,\psi}(\mathbf{R}_+)$ ,  $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$ , we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy < k(\lambda_1) \|f\|_{p,\phi} \|g\|_{q,\psi}, \quad (1.5)$$

where, the constant factor  $k(\lambda_1)$  is the best possible. Moreover, if  $k_\lambda(x, y)$  keeps finite and  $k_\lambda(x, y) x^{\lambda_1-1} (k_\lambda(x, y) y^{\lambda_2-1})$  is decreasing with respect to  $x > 0$  ( $y > 0$ ), then for  $a_m, b_n \geq 0$ ,

$$a \in l_{p,\phi} = \left\{ a; \|a\|_{p,\phi} := \left( \sum_{n=1}^\infty \phi(n) |a_n|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^{\infty} \in l_{q,\Psi}$ ,  $\|a\|_{p,\Phi}, \|b\|_{q,\Psi} > 0$ , we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_{\lambda}(m, n) a_m b_n < k(\lambda_1) \|a\|_{p,\Phi} \|b\|_{q,\Psi}, \quad (1.6)$$

where, the constant factor  $k(\lambda_1)$  is still the best possible.

For  $0 < \lambda_1, \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda$ , we set

$$k_{\lambda}(x, y) = \frac{1}{(x+y)^{\lambda}} \quad ((x, y) \in \mathbf{R}_+^2).$$

Then by (1.6), we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \|a\|_{p,\Phi} \|b\|_{q,\Psi}, \quad (1.7)$$

where, the constant  $B(\lambda_1, \lambda_2)$  is the best possible, and

$$B(u, v) = \int_0^{\infty} \frac{1}{(1+t)^{u+v}} t^{u-1} dt \quad (u, v > 0)$$

is the beta function. Clearly, for  $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ , inequality (1.7) reduces to (1.2).

In 2015, by adding some conditions, Yang [7] extended (1.7) and (1.4) as follows:

$$\begin{aligned} & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(U_m + V_n)^{\lambda}} \\ & < B(\lambda_1, \lambda_2) \left[ \sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{\mu_m^{p-1}} \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1} b_n^q}{\nu_n^{q-1}} \right]^{\frac{1}{q}}, \end{aligned} \quad (1.8)$$

where, the constant  $B(\lambda_1, \lambda_2)$  is still the best possible.

Some other results including multidimensional Hilbert-type inequalities are provided in [8]-[30].

Related to the topic of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are the best possible. However, Yang [31] established a result with the kernel  $\frac{1}{(1+nx)^{\lambda}}$  by introducing a variable and proved that the constant factor is the best possible. In 2011 Yang [32] proved the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor  $B(\lambda_1, \lambda_2)$ :

$$\int_0^{\infty} f(x) \left[ \sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}} \right] dx < B(\lambda_1, \lambda_2) \|f\|_{p,\Phi} \|a\|_{q,\Psi}, \quad (1.9)$$

where,  $\lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda$ . Zhong et al ([33]-[39]) investigated several half-discrete Hilbert-type inequalities with particular kernels. Applying the method of weight functions, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree  $-\lambda \in \mathbf{R}$  and a best constant factor  $k(\lambda_1)$  is obtained as follows:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} k_{\lambda}(x, n) a_n dx < k(\lambda_1) \|f\|_{p,\Phi} \|a\|_{q,\Psi}, \quad (1.10)$$

which is an extension of (1.9) (cf. [40]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [41]. In 2012-2014, Yang et al. published three books [42], [43] and [44] extensively presenting the framework of half-discrete Hilbert-type inequalities.

In this paper, by the method of weight coefficients, techniques of real analysis and Hermite-Hadamard's inequality, a half-discrete Hardy-Hilbert-type inequality related to the kernel of the hyperbolic cosecant function with a best possible constant factor expressed by the extended Riemann-zeta function is proved, which is an extension of (1.10) for  $\lambda = 0$  in the following particular kernel:

$$k_0(x, n) = \frac{\csc h(\rho(\frac{n}{x})^\gamma)}{e^{\alpha(\frac{n}{x})^\gamma}} (\rho > \max\{0, -\alpha\}, 0 < \gamma < 1).$$

Furthermore, the more accurate equivalent forms, the operator expressions with the norm, the reverses and some particular cases are also considered.

## 2 Some Lemmas

In the sequel, we shall assume that  $v_n > 0$  ( $n \in \mathbf{N}$ ),  $\{v_n\}_{n=1}^\infty$  is decreasing,  $V_n = \sum_{j=1}^n v_j$ ,  $\mu(t)$  is a positive continuous function in  $\mathbf{R}_+ = (0, \infty)$ ,

$$U(0) := 0; \quad U(x) := \int_0^x \mu(t) dt < \infty (x \in (0, \infty)),$$

$$v(t) := v_n, \quad t \in (n-1, n] \quad (n \in \mathbf{N}),$$

and

$$V(0) := 0; \quad V(y) := \int_0^y v(t) dt (y \in (0, \infty)),$$

$$p \neq 0, 1, \frac{1}{p} + \frac{1}{q} = 1, \delta \in \{-1, 1\}, \beta \leq \frac{v_1}{2}, f(x), a_n \geq 0 (x \in \mathbf{R}_+, n \in \mathbf{N}),$$

$$\|f\|_{p, \Phi_\delta} = \left( \int_0^\infty \Phi_\delta(x) f^p(x) dx \right)^{\frac{1}{p}},$$

$$\|a\|_{q, \Psi} = \left( \sum_{n=1}^\infty \Psi_\beta(n) b_n^q \right)^{\frac{1}{q}},$$

where,

$$\Phi_\delta(x) \quad : \quad = \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} (x \in \mathbf{R}_+),$$

$$\Psi_\beta(n) \quad : \quad = \frac{(V_n - \beta)^{q(1-\sigma)-1}}{v_{n+1}^{q-1}} (n \in \mathbf{N}).$$

**Lemma 2.1.** *If  $a \in \mathbf{R}$ ,  $f(x)$  is continuous in  $[a - \frac{1}{2}, a + \frac{1}{2}]$ ,  $f'(x)$  is strictly increasing in  $(a - \frac{1}{2}, a)$  and  $(a, a + \frac{1}{2})$  respectively, as well as*

$$\lim_{x \rightarrow a-} f'(x) = f'(a-0) \leq f'(a+0) = \lim_{x \rightarrow a+} f'(x),$$

then  $f(x)$  is strictly convex in  $[a - \frac{1}{2}, a + \frac{1}{2}]$ , and we have the following Hermite-Hadamard's inequality (cf. [48]):

$$f(a) < \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x) dx. \quad (2.1)$$

*Proof.* Since  $f'(a-0) (\leq f'(a+0))$  is finite, we define a function  $g(x)$  as follows:

$$g(x) := f'(a-0)(x-a) + f(a), x \in [a - \frac{1}{2}, a + \frac{1}{2}].$$

In view of  $f'(x)$  being strictly increasing in  $(a - \frac{1}{2}, a)$ , then for  $x \in (a - \frac{1}{2}, a)$ ,

$$(f(x) - g(x))' = f'(x) - f'(a-0) < 0.$$

Since  $f(a) - g(a) = 0$ , it follows that  $f(x) - g(x) > 0$ ,  $x \in (a - \frac{1}{2}, a)$ . Similarly, we can obtain  $f(x) - g(x) > 0$ ,  $x \in (a, a + \frac{1}{2})$ . Hence,  $f(x)$  is strictly convex in  $[a - \frac{1}{2}, a + \frac{1}{2}]$ , and therefore

$$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x) dx > \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} g(x) dx = f(a),$$

namely, (2.1) follows. □

**Example 2.2.** For  $\rho > \max\{0, -\alpha\}$ ,  $0 < \gamma < \sigma \leq 1$ ,

$$\csc h(u) = \frac{2}{e^u - e^{-u}} \quad (u > 0)$$

is called hyperbolic cosecant function (cf. [45]), we set

$$h(t) = \frac{\csc h(\rho t^\gamma)}{e^{\alpha t^\gamma}} = \frac{2}{e^{(\alpha+\rho)t^\gamma}(1 - e^{-2\rho t^\gamma})} \quad (t \in \mathbf{R}_+).$$

(i) Setting  $u = \rho t^\gamma$ , we find

$$\begin{aligned} k(\sigma) &:= \int_0^\infty \frac{\csc h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\ &= \frac{1}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{\csc h(u)}{e^{\frac{\alpha}{\rho} u}} u^{\frac{\sigma}{\gamma}-1} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-\frac{\alpha}{\rho} u} u^{\frac{\sigma}{\gamma}-1}}{e^u - e^{-u}} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-(\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1}}{1 - e^{-2u}} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \sum_{k=0}^\infty e^{-(2k+\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1} du. \end{aligned}$$

By Lebesgue's term by term theorem (cf. [45]), setting  $v = (2k + \frac{\alpha}{\rho} + 1)u$ , we have

$$\begin{aligned}
k(\sigma) &= \int_0^\infty \frac{\csc h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\
&= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \int_0^\infty e^{-(2k + \frac{\alpha}{\rho} + 1)u} u^{\frac{\sigma}{\gamma}-1} du \\
&= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \frac{1}{(2k + \frac{\alpha}{\rho} + 1)^{\sigma/\gamma}} \int_0^\infty e^{-v} v^{\frac{\sigma}{\gamma}-1} dv \\
&= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \sum_{k=0}^\infty \frac{1}{(k + \frac{\alpha+\rho}{2\rho})^{\sigma/\gamma}} \\
&= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta\left(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}\right) \in \mathbf{R}_+, \tag{2.2}
\end{aligned}$$

where

$$\zeta(s, a) := \sum_{k=0}^\infty \frac{1}{(k+a)^s} \quad (Re(s) > 1, a > 0)$$

is called the extended Riemann-zeta function (also known as the Hurwitz zeta function)<sup>1</sup>, and

$$\Gamma(y) := \int_0^\infty e^{-v} v^{y-1} dv \quad (y > 0)$$

is called Gamma function (cf. [46]).

In particular, for  $\alpha = \rho$ , we have

$$h(t) = \frac{\csc h(\rho t^\gamma)}{e^{\rho t^\gamma}} \quad \text{and} \quad k(\sigma) = k_1(\sigma) := \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta\left(\frac{\sigma}{\gamma}\right).$$

In this case, for  $\gamma = \frac{\sigma}{2}$ , we have

$$h(t) = \frac{\csc h(\rho t^{\sigma/2})}{e^{\rho t^{\sigma/2}}} \quad \text{and} \quad k(\sigma) = \frac{\pi^2}{6\sigma\rho^2}.$$

(ii) We obtain for  $u > 0$  that

$$\frac{1}{1-e^{-2u}} > 0, \quad \left(\frac{1}{1-e^{-2u}}\right)' = -\frac{2e^{-2u}}{(1-e^{-2u})^2} < 0,$$

and

$$\left(\frac{1}{1-e^{-2u}}\right)'' = \frac{4e^{-2u}}{(1-e^{-2u})^2} + \frac{8e^{-4u}}{(1-e^{-2u})^3} > 0.$$

(iii) If  $g(u) > 0, g'(u) < 0, g''(u) > 0$ , then for  $0 < \gamma \leq 1$ , we find that  $g(\rho t^\gamma) > 0$ ,  $\frac{d}{dt}g(\rho t^\gamma) = \rho\gamma t^{\gamma-1}g'(\rho t^\gamma) < 0$ , and

$$\frac{d^2}{dt^2}g(\rho t^\gamma) = \rho\gamma(\gamma-1)t^{\gamma-2}g'(\rho t^\gamma) + (\rho\gamma t^{\gamma-1})^2g''(\rho t^\gamma) > 0.$$

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<sup>1</sup>Clearly  $\zeta(s, 1) = \zeta(s)$ , where  $\zeta(s)$  is the Riemann-zeta function.

Then we find that for  $y \in (n - \frac{1}{2}, n)$ ,

$$g(V(y) - \beta) > 0, \frac{d}{dy}g(V(y) - \beta) = g'(V(y) - \beta)v_n < 0,$$

and

$$\frac{d^2}{dy^2}g(V(y) - \beta) = g''(V(y) - \beta)v_n^2 > 0 \quad (n \in \mathbf{N});$$

for  $y \in (n, n + \frac{1}{2})$ ,

$$g(V(y) - \beta) > 0, \frac{d}{dy}g(V(y) - \beta) = g'(V(y) - \beta)v_{n+1} < 0,$$

and

$$\frac{d^2}{dy^2}g(V(y) - \beta) = g''(V(y) - \beta)v_{n+1}^2 > 0 \quad (n \in \mathbf{N}).$$

If  $g_1(u) > 0, g'_1(u) < 0, g''_1(u) > 0, g_2(u) > 0, g'_2(u) \leq 0, g''_2(u) \geq 0$ , then we find for  $u > 0$  that

$$g_1(u)g_2(u) > 0, (g_1(u)g_2(u))' = g'_1(u)g_2(u) + g_1(u)g'_2(u) < 0,$$

and

$$(g_1(u)g_2(u))'' = g''_1(u)g_2(u) + 2g'_1(u)g'_2(u) + g_1(u)g''_2(u) > 0.$$

(iv) For  $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1$ , we have

$$h(t) > 0, h'(t) < 0, h''(t) > 0, \text{ with } k(\sigma) \in \mathbf{R}_+,$$

and then for  $c > 0, \beta \leq \frac{v_1}{2}, y \geq \frac{1}{2}, n \in \mathbf{N}$ , we have

$$h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1} > 0, \frac{d}{dy}[h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}] < 0,$$

and

$$\frac{d^2}{dy^2}[h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}] > 0 \quad (y \in (n - \frac{1}{2}, n) \cup (n, n + \frac{1}{2})).$$

Setting  $f(y) = h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}$ , it follows that  $f'(y)(< 0)$  is strictly increasing in  $(n - \frac{1}{2}, n)$  and

$$\begin{aligned} \lim_{x \rightarrow n-} f'(y) &= f'(n-0) = [ch'(c(V_n - \beta))(V_n - \beta)^{\sigma-1} \\ &\quad + (\sigma - 1)h(c(V_n - \beta))(V_n - \beta)^{\sigma-2}]v_n. \end{aligned}$$

In the same way, for  $x \in (n, n + \frac{1}{2})$ , we find that  $f'(y)(< 0)$  is strictly increasing and

$$\begin{aligned} \lim_{x \rightarrow n+} f'(y) &= f'(n+0) = [ch'(c(V_n - \beta))(V_n - \beta)^{\sigma-1} \\ &\quad + (\sigma - 1)h(c(V_n - \beta))(V_n - \beta)^{\sigma-2}]v_{n+1}. \end{aligned}$$

In view of  $v_{n+1} \leq v_n$ , it follows that

$$\lim_{x \rightarrow n+} f'(x) = f'(n+0) \geq f'(n-0) = \lim_{x \rightarrow n-} f'(x).$$

Then by (2.1), for  $n \in \mathbf{N}$ , we have

$$f(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(y) dy = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1} dy. \quad (2.3)$$

**Lemma 2.3.** *If  $g(t) (> 0)$  is a strictly decreasing continuous function in  $(\frac{1}{2}, \infty)$ , which is strictly convex satisfying*

$$\int_{\frac{1}{2}}^{\infty} g(t) dt \in \mathbf{R}_+,$$

*then we have*

$$\int_1^{\infty} g(t) dt < \sum_{n=1}^{\infty} g(n) < \int_{\frac{1}{2}}^{\infty} g(t) dt. \quad (2.4)$$

*Proof.* By (2.1) and the decreasing property, we have

$$\int_n^{n+1} g(t) dt < \int_n^{n+1} g(n) dt = g(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t) dt \quad (n \in \mathbf{N}),$$

and for  $n_0 \in \mathbf{N}$ , it follows that

$$\begin{aligned} \int_1^{n_0+1} g(t) dt &< \sum_{n=1}^{n_0} g(n) < \sum_{n=1}^{n_0} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t) dt = \int_{\frac{1}{2}}^{n_0+\frac{1}{2}} g(t) dt, \\ \int_{n_0+1}^{\infty} g(t) dt &\leq \sum_{n=n_0+1}^{\infty} g(n) \leq \int_{n_0+\frac{1}{2}}^{\infty} g(t) dt < \infty. \end{aligned}$$

Hence, we obtain (2.4).  $\square$

**Lemma 2.4.** *If  $\rho > \max\{0, -\alpha\}$ ,  $0 < \gamma < \sigma \leq 1$ , define the following weight coefficients:*

$$\omega_{\delta}(\sigma, x) : = \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} \frac{U^{\delta\sigma}(x) v_{n+1}}{(V_n - \beta)^{1-\sigma}}, \quad x \in \mathbf{R}_+, \quad (2.5)$$

$$\varpi_{\delta}(\sigma, n) : = \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} \frac{(V_n - \beta)^{\sigma} \mu(x)}{U^{1-\delta\sigma}(x)} dx, \quad n \in \mathbf{N}. \quad (2.6)$$

*Then, we have the following inequalities:*

$$\omega_{\delta}(\sigma, x) < k(\sigma) \quad (x \in \mathbf{R}_+), \quad (2.7)$$

$$\varpi_{\delta}(\sigma, n) \leq k(\sigma) \quad (n \in \mathbf{N}), \quad (2.8)$$

where,  $k(\sigma)$  is indicated by (2.2).



*Proof.* Since  $V_n = V(n)$ , and for  $t \in (n - \frac{1}{2}, n)$ ,

$$v_{n+1} \leq v_n = V'(t);$$

for  $t \in (n, n + \frac{1}{2})$ ,

$$v_{n+1} = V'(t),$$

by (2.3) (for  $c = U^\delta(x)$ ), we have

$$\begin{aligned} & \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{U^{\delta\sigma}(x)}{(V_n - \beta)^{1-\sigma}} \\ &= \frac{\csc h(\rho U^{\delta\gamma}(x)(V(n) - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(n) - \beta)^\gamma}} \frac{U^{\delta\sigma}(x)}{(V(n) - \beta)^{1-\sigma}} \\ &< \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t) - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t) - \beta)^\gamma}} \frac{U^{\delta\sigma}(x)}{(V(t) - \beta)^{1-\sigma}} dt \quad (n \in \mathbf{N}), \\ \omega_\delta(\sigma, x) &< \sum_{n=1}^{\infty} v_{n+1} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t) - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t) - \beta)^\gamma}} \frac{U^{\delta\sigma}(x) dt}{(V(t) - \beta)^{1-\sigma}} \\ &\leq \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t) - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t) - \beta)^\gamma}} \frac{U^{\delta\sigma}(x) V'(t)}{(V(t) - \beta)^{1-\sigma}} dt \\ &= \int_{\frac{1}{2}}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t) - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t) - \beta)^\gamma}} \frac{U^{\delta\sigma}(x) V'(t)}{(V(t) - \beta)^{1-\sigma}} dt. \end{aligned}$$

Setting  $u = U^\delta(x)(V(t) - \beta)$ , by (2.2), we obtain

$$\begin{aligned} \omega_\delta(\sigma, x) &< \int_{U^\delta(x)(\frac{1}{2}-\beta)}^{U^\delta(x)V(\infty)} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} \frac{U^{\delta\sigma}(x) U^{-\delta}(x)}{(u U^{-\delta}(x))^{1-\sigma}} du \\ &\leq \int_0^\infty \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du = k(\sigma). \end{aligned}$$

Hence, (2.7) follows.

Setting  $u = (V_n - \beta)U^\delta(x)$  in (2.6), we find  $du = \delta(V_n - \beta)U^{\delta-1}(x)\mu(x)dx$  and

$$\begin{aligned} \mathfrak{W}_\delta(\sigma, n) &= \frac{1}{\delta} \int_{(V_n - \beta)U^\delta(0)}^{(V_n - \beta)U^\delta(\infty)} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} \frac{(V_n - \beta)^{\sigma-1} [(V_n - \beta)^{-1}u]^{\frac{1}{\delta}-1}}{[(V_n - \beta)^{-1}u]^{\frac{1}{\delta}-\sigma}} du \\ &= \frac{1}{\delta} \int_{(V_n - \beta)U^\delta(0)}^{(V_n - \beta)U^\delta(\infty)} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du. \end{aligned}$$

If  $\delta = 1$ , then

$$\mathfrak{W}_1(\sigma, n) = \int_0^{(V_n - \beta)U(\infty)} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \leq \int_0^\infty \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du;$$

if  $\delta = -1$ , then

$$\mathfrak{W}_{-1}(\sigma, n) = - \int_\infty^{(V_n - \beta)U^{-1}(\infty)} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \leq \int_0^\infty \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du.$$

Hence, by (2.2), we have (2.8).  $\square$

*Remark 2.5.* We do not need the condition of  $\sigma \leq 1$  in obtaining (2.8). If  $U(\infty) = \infty$ , then we have

$$\mathfrak{w}_\delta(\sigma, n) = k(\sigma) \quad (n \in \mathbf{N}). \quad (2.9)$$

For example, we set  $\mu(t) = \frac{1}{(1+t)^a}$  ( $t > 0; 0 \leq a \leq 1$ ), then for  $x \geq 0$ , we find

$$U(x) = \int_0^x \frac{dt}{(1+t)^a} = \begin{cases} \frac{(1+x)^{1-a}-1}{1-a}, & 0 \leq a < 1 \\ \ln(1+x), & a = 1 \end{cases} < \infty,$$

$$U(0) = 0 \text{ and } U(\infty) = \int_0^\infty \frac{dt}{(1+t)^a} = \infty.$$

**Lemma 2.6.** *If  $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1, V(\infty) = \infty$ , then, (i) for  $x \in \mathbf{R}_+$ , we have*

$$k(\sigma)(1 - \theta_\delta(\sigma, x)) < \omega_\delta(\sigma, x), \quad (2.10)$$

where,

$$\begin{aligned} \theta_\delta(\sigma, x) &:= \frac{1}{k(\sigma)} \int_0^{U^\delta(x)(v_1-\beta)} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \\ &= O((U(x))^{\frac{\delta}{2}(\sigma-\gamma)}) \in (0, 1); \end{aligned}$$

(ii) for any  $b > 0$ , we have

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1+b}} = \frac{1}{b} \left[ \frac{1}{(v_1 - \beta)^b} + bO(1) \right]. \quad (2.11)$$

*Proof.* By (2.4), we find

$$\begin{aligned} \omega_\delta(\sigma, x) &> \sum_{n=1}^{\infty} v_{n+1} \int_n^{n+1} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t) - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t) - \beta)^\gamma}} \frac{U^{\delta\sigma}(x) dt}{(V(t) - \beta)^{1-\sigma}} \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t) - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t) - \beta)^\gamma}} \frac{U^{\delta\sigma}(x) V'(t)}{(V(t) - \beta)^{1-\sigma}} dt \\ &= \int_1^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t) - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t) - \beta)^\gamma}} \frac{U^{\delta\sigma}(x) V'(t)}{(V(t) - \beta)^{1-\sigma}} dt. \end{aligned}$$

Setting  $u = U^\delta(x)(V(t) - \beta)$ , in view of  $V(\infty) = \infty$ , by (2.2), we find

$$\begin{aligned} \omega_\delta(\sigma, x) &> \int_{U^\delta(x)(V(1)-\beta)}^\infty \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \\ &= k(\sigma) - \int_0^{U^\delta(x)(v_1-\beta)} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \\ &= k(\sigma)(1 - \theta_\delta(\sigma, x)). \end{aligned}$$

Since

$$F(u) = \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}}$$

is continuous in  $(0, \infty)$  satisfying  $u^{\frac{1}{2}(\sigma+\gamma)}F(u) \rightarrow 0$  ( $u \rightarrow 0^+$ ), and  $u^{\frac{1}{2}(\sigma+\gamma)}F(u) \rightarrow 0$  ( $u \rightarrow \infty$ ), there exists a constant  $L > 0$ , such that  $u^{\frac{1}{2}(\sigma+\gamma)}F(u) \leq L$ , namely,

$$\frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} \leq L u^{\frac{1}{2}(\sigma+\gamma)} \quad (u \in (0, \infty)).$$

Hence we find

$$\begin{aligned} 0 &< \theta_\delta(\sigma, x) \leq \frac{L}{k(\sigma)} \int_0^{U^\delta(x)(v_1-\beta)} u^{\frac{1}{2}(\sigma-\gamma)-1} du \\ &= \frac{2L[U^\delta(x)(v_1-\beta)]^{\frac{1}{2}(\sigma-\gamma)}}{k(\sigma)(\sigma-\gamma)} \quad (x \in \mathbf{R}_+), \end{aligned}$$

and then (2.10) follows.

For  $b > 0$ , we find

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1+b}} &< \frac{v_2}{(V_1 - \beta)^{1+b}} + \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{V'(x)}{(V(x) - \beta)^{1+b}} dx \\ &= \frac{v_2}{(v_1 - \beta)^{1+b}} + \int_{\frac{3}{2}}^{\infty} \frac{V'(x)}{(V(x) - \beta)^{1+b}} dx \\ &= \frac{v_2}{(v_1 - \beta)^{1+b}} + \int_{v_1+\frac{1}{2}v_2-\beta}^{\infty} \frac{du}{u^{1+b}} \\ &\leq \frac{1}{b} \left[ \frac{1}{(v_1 - \beta)^b} + b \frac{v_2}{(v_1 - \beta)^{1+b}} \right], \\ \sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1+b}} &= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{v_{n+1}}{(V(n) - \beta)^{1+b}} dx > \sum_{n=1}^{\infty} \int_n^{n+1} \frac{V'(x) dx}{(V(x) - \beta)^{1+b}} \\ &= \int_1^{\infty} \frac{V'(x) dx}{(V(x) - \beta)^{1+b}} = \frac{1}{b(v_1 - \beta)^b}. \end{aligned}$$

Hence we have (2.11).  $\square$

**Note.** For example,  $v_n = \frac{1}{n^a}$  ( $n \in \mathbf{N}; 0 \leq a \leq 1$ ) satisfies the condition that  $v_n > 0$  ( $n \in \mathbf{N}$ ),  $\{v_n\}_{n=1}^{\infty}$  is decreasing, and  $V(\infty) = \infty$ .

### 3 Main Results and Operator Expressions

**Theorem 3.1.** If  $\rho > \max\{0, -\alpha\}$ ,  $0 < \gamma < \sigma \leq 1$ ,  $k(\sigma)$  is indicated by (2.2), then for  $p > 1$ ,  $0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities:

$$I := \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n f(x) dx < k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \quad (3.1)$$

$$\begin{aligned} J_1 &:= \sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p \\ &< k(\sigma) \|f\|_{p, \Phi_\delta}^p, \end{aligned} \quad (3.2)$$

$$\begin{aligned}
J_2 &:= \left\{ \int_0^\infty \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} \\
&< k(\sigma) \|a\|_{q, \Psi_\beta}.
\end{aligned} \tag{3.3}$$

*Proof.* By the weighted Hölder inequality (cf. [48]), we have

$$\begin{aligned}
&\left[ \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p \\
&= \left[ \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \right. \\
&\quad \times \frac{U^{\frac{1-\delta\sigma}{q}}(x) f(x)}{(V_n - \beta)^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)} \cdot \left. \frac{(V_n - \beta)^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)}{U^{\frac{1-\delta\sigma}{q}}(x)} dx \right]^p \\
&\leq \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \left[ \frac{U^{\frac{p(1-\delta\sigma)}{q}}(x) f^p(x)}{(V_n - \beta)^{1-\sigma} \mu^{\frac{p}{q}}(x)} \right] dx \\
&\quad \times \left[ \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{(V_n - \beta)^{(1-\sigma)(p-1)} \mu(x)}{U^{1-\delta\sigma}(x)} dx \right]^{p-1} \\
&= \frac{(\mathfrak{W}_\delta(\sigma, n))^{p-1}}{(V_n - \beta)^{p\sigma-1} v_{n+1}} \\
&\quad \times \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_{n+1} f^p(x)}{(V_n - \beta)^{1-\sigma} \mu^{p-1}(x)} dx.
\end{aligned} \tag{3.4}$$

In view of (2.8) and the Lebesgue term by term integration theorem (cf. [47]), we find

$$\begin{aligned}
J_1 &\leq (k(\sigma))^{\frac{1}{q}} \left[ \sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_{n+1}}{(V_n - \beta)^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}} \\
&= (k(\sigma))^{\frac{1}{q}} \left[ \int_0^\infty \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_{n+1}}{(V_n - \beta)^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}} \\
&= (k(\sigma))^{\frac{1}{q}} \left[ \int_0^\infty \omega_\delta(\sigma, x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}}.
\end{aligned} \tag{3.5}$$

Then by (2.7), we derive (3.2).

By Hölder's inequality (cf. [48]), we have

$$\begin{aligned}
I &= \sum_{n=1}^\infty \left[ \frac{v_{n+1}^{\frac{1}{p}}}{(V_n - \beta)^{\frac{1}{p}-\sigma}} \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right] \\
&= \left[ \frac{(V_n - \beta)^{\frac{1}{p}-\sigma} a_n}{v_{n+1}^{\frac{1}{p}}} \right] \leq J_1 \|a\|_{q, \Psi_\beta}.
\end{aligned} \tag{3.6}$$

Then by (3.2), we obtain (3.1). On the other hand, assuming that (3.1) is valid, we set

$$a_n := \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^{p-1}, \quad n \in \mathbf{N}.$$

Then we find  $J_1^p = \|a\|_{q, \Psi_\beta}^q$ . If  $J_1 = 0$ , then (3.2) is trivially valid; if  $J_1 = \infty$ , then (3.2) is still not valid. Suppose that  $0 < J_1 < \infty$ . By (3.1), we have

$$\begin{aligned} \|a\|_{q, \Psi_\beta}^q &= J_1^p = I < k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \\ \|a\|_{q, \Psi_\beta}^{q-1} &= J_1 < k(\sigma) \|f\|_{p, \Phi_\delta}, \end{aligned}$$

and then (3.2) follows, which is equivalent to (3.1).

Still by the weighted Hölder inequality (cf. [48]), we have

$$\begin{aligned} & \left[ \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q \\ &= \left[ \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \cdot \frac{U^{\frac{1-\delta\sigma}{q}}(x) v_{n+1}^{\frac{1}{p}}}{(V_n - \beta)^{\frac{1-\sigma}{p}}} \cdot \frac{(V_n - \beta)^{\frac{1-\sigma}{p}} a_n}{U^{\frac{1-\delta\sigma}{q}}(x) v_{n+1}^{\frac{1}{p}}} \right]^q \\ &\leq \left[ \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_{n+1}}{(V_n - \beta)^{1-\sigma}} \right]^{q-1} \\ &\quad \times \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{(V_n - \beta)^{\frac{q(1-\sigma)}{p}}}{U^{1-\delta\sigma}(x) v_{n+1}^{q-1}} a_n^q \\ &= \frac{(\omega_\delta(\sigma, x))^{q-1}}{U^{q\delta\sigma-1}(x) \mu(x)} \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{(V_n - \beta)^{(1-\sigma)(q-1)} \mu(x)}{U^{1-\delta\sigma}(x) v_{n+1}^{q-1}} a_n^q. \quad (3.7) \end{aligned}$$

Then by (2.7) and the Lebesgue term by term integration theorem (cf. [47]), it follows that

$$\begin{aligned} J_2 &< (k(\sigma))^{\frac{1}{p}} \left[ \int_0^\infty \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{(V_n - \beta)^{(1-\sigma)(q-1)} \mu(x)}{U^{1-\delta\sigma}(x) v_{n+1}^{q-1}} a_n^q dx \right]^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left[ \sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{(V_n - \beta)^{(1-\sigma)(q-1)} \mu(x)}{U^{1-\delta\sigma}(x) v_{n+1}^{q-1}} a_n^q dx \right]^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left[ \sum_{n=1}^\infty \overline{\omega}_\delta(\sigma, n) \frac{(V_n - \beta)^{q(1-\sigma)-1}}{v_{n+1}^{q-1}} a_n^q \right]^{\frac{1}{q}}. \quad (3.8) \end{aligned}$$

Then by (2.8), we derive (3.3).

By Hölder's inequality (cf. [48]), we have

$$\begin{aligned} I &= \int_0^\infty \left( \frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x) \right) \left[ \frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right] dx \\ &\leq \|f\|_{p, \Phi_\delta} J_2. \quad (3.9) \end{aligned}$$

Then by (3.3), we obtain (3.1). On the other hand, assuming that (3.3) is valid, we set

$$f(x) := \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^{q-1}, \quad x \in \mathbf{R}_+.$$

Then we find  $J_2^q = \|f\|_{p, \Phi_\delta}^p$ . If  $J_2 = 0$ , then (3.3) is trivially valid; if  $J_2 = \infty$ , then (3.3) remains impossible. Suppose that  $0 < J_2 < \infty$ . By (3.1), we have

$$\begin{aligned} \|f\|_{p, \Phi_\delta}^p &= J_2^q = I < k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \\ \|f\|_{p, \Phi_\delta}^{p-1} &= J_2 < k(\sigma) \|a\|_{q, \Psi_\beta}, \end{aligned}$$

and then (3.3) follows, which is equivalent to (3.1).

Therefore, (3.1), (3.2) and (3.3) are equivalent.  $\square$

**Theorem 3.2.** *With the assumptions of Theorem 3.1, if  $U(\infty) = V(\infty) = \infty$ , then the constant factor  $k(\sigma)$  in (3.1), (3.2) and (3.3) is the best possible.*

*Proof.* For  $\varepsilon \in (0, \frac{q(\sigma-\gamma)}{2})$ , we set  $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q}$ , and  $\tilde{f} = \tilde{f}(x)$ ,  $x \in \mathbf{R}_+$ ,  $\tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$ ,

$$\tilde{f}(x) = \begin{cases} U^{\delta(\tilde{\sigma}+\varepsilon)-1}(x)\mu(x), & 0 < x^\delta \leq 1 \\ 0, & x^\delta > 0 \end{cases}, \quad (3.10)$$

$$\tilde{a}_n = (V_n - \beta)^{\tilde{\sigma}-1} v_{n+1} = (V_n - \beta)^{\sigma - \frac{\varepsilon}{q} - 1} v_{n+1}, \quad n \in \mathbf{N}. \quad (3.11)$$

Then for  $\delta = \pm 1$ , since  $U(\infty) = \infty$ , we find

$$\int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx = \frac{1}{\varepsilon} U^{\delta\varepsilon}(1). \quad (3.12)$$

By (2.11), (3.12) and (2.10), we obtain

$$\begin{aligned} \|\tilde{f}\|_{p, \Phi_\delta} \|\tilde{a}\|_{q, \Psi_\beta} &= \left( \int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\mu(x) dx}{U^{1-\delta\varepsilon}(x)} \right)^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1+\varepsilon}} \right]^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} U^{\frac{\delta\varepsilon}{p}}(1) \left[ \frac{1}{(v_1 - \beta)^\varepsilon} + \varepsilon O(1) \right]^{\frac{1}{q}}, \end{aligned} \quad (3.13)$$

$$\begin{aligned}
\tilde{I} &: = \int_0^\infty \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \tilde{a}_n \tilde{f}(x) dx \\
&= \int_{\{x>0; 0<x^\delta \leq 1\}} \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{(V_n - \beta)^{\tilde{\sigma}-1} v_{n+1} \mu(x)}{U^{1-\delta(\tilde{\sigma}+\varepsilon)}(x)} dx \\
&= \int_{\{x>0; 0<x^\delta \leq 1\}} \omega_\delta(\tilde{\sigma}, x) \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx \\
&\geq k(\tilde{\sigma}) \int_{\{x>0; 0<x^\delta \leq 1\}} (1 - \theta_\delta(\tilde{\sigma}, x)) \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx \\
&= k(\tilde{\sigma}) \int_{\{x>0; 0<x^\delta \leq 1\}} (1 - O((U(x))^{\delta \frac{\sigma - \frac{\varepsilon}{q} - \gamma}})) \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx \\
&= k(\tilde{\sigma}) \left[ \int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\mu(x) dx}{U^{1-\delta\varepsilon}(x)} - \int_{\{x>0; 0<x^\delta \leq 1\}} O\left(\frac{\mu(x)}{U^{1-\delta(\varepsilon + \frac{\sigma - \frac{\varepsilon}{q} - \gamma}{2})}(x)}\right) dx \right] \\
&= \frac{1}{\varepsilon} k(\sigma - \frac{\varepsilon}{q}) (U^{\delta\varepsilon}(1) - \varepsilon O_1(1)).
\end{aligned}$$

If there exists a positive constant  $K \leq k(\sigma)$ , such that (3.1) is valid when replacing  $k(\sigma)$  to  $K$ , then in particular, by the Lebesgue term by term integration theorem, we have

$$\varepsilon \tilde{I} < \varepsilon K \|\tilde{f}\|_{p, \Phi_\delta} \|\tilde{a}\|_{q, \Psi_\beta},$$

namely,

$$k(\sigma - \frac{\varepsilon}{q}) (U^{\delta\varepsilon}(1) - \varepsilon O_1(1)) < K \cdot U^{\frac{\delta\varepsilon}{p}}(1) \left[ \frac{1}{(v_1 - \beta)^\varepsilon} + \varepsilon O(1) \right]^{\frac{1}{q}}.$$

It follows that  $k(\sigma) \leq K(\varepsilon \rightarrow 0^+)$ . Hence,  $K = k(\sigma)$  is the best possible constant factor of (3.1).

The constant factor  $k(\sigma)$  in (3.2) ((3.3)) is still the best possible. Otherwise, we would reach a contradiction by (3.6) ((3.9)) that the constant factor in (3.1) is not the best possible.  $\square$

For  $p > 1$ , we obtain

$$\Psi_\beta^{1-p}(n) = \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \quad (n \in \mathbf{N}), \quad \Phi_\delta^{1-q}(x) = \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \quad (x \in \mathbf{R}_+),$$

and define the following real normed spaces:

$$\begin{aligned}
L_{p, \Phi_\delta}(\mathbf{R}_+) &= \{f; f = f(x), x \in \mathbf{R}_+, \|f\|_{p, \Phi_\delta} < \infty\}, \\
l_{q, \Psi_\beta} &= \{a; a = \{a_n\}_{n=1}^\infty, \|a\|_{q, \Psi_\beta} < \infty\}, \\
L_{q, \Phi_\delta^{1-q}}(\mathbf{R}_+) &= \{h; h = h(x), x \in \mathbf{R}_+, \|h\|_{q, \Phi_\delta^{1-q}} < \infty\}, \\
l_{p, \Psi_\beta^{1-p}} &= \{c; c = \{c_n\}_{n=1}^\infty, \|c\|_{p, \Psi_\beta^{1-p}} < \infty\}.
\end{aligned}$$

Assuming that  $f \in L_{p, \Phi_\delta}(\mathbf{R}_+)$  and setting

$$c = \{c_n\}_{n=1}^\infty, \quad c_n := \int_0^\infty \frac{\csc h(\rho [U^\delta(x)(V_n - \beta)]^\gamma)}{e^{\alpha [U^\delta(x)(V_n - \beta)]^\gamma}} f(x) dx, \quad n \in \mathbf{N},$$

we can rewrite (3.2) as

$$\|c\|_{p, \Psi_\beta^{1-p}} < k(\sigma) \|f\|_{p, \Phi_\delta} < \infty,$$

namely,  $c \in l_{p, \Psi_\beta^{1-p}}$ .

**Definition 3.3.** Define a half-discrete Hardy-Hilbert-type operator  $T_1 : L_{p, \Phi_\delta}(\mathbf{R}_+) \rightarrow l_{p, \Psi_\beta^{1-p}}$  as follows: For any  $f \in L_{p, \Phi_\delta}(\mathbf{R}_+)$ , there exists a unique representation  $T_1 f = c \in l_{p, \Psi_\beta^{1-p}}$ . Define the formal inner product of  $T_1 f$  and  $a = \{a_n\}_{n=1}^\infty \in l_{q, \Psi_\beta}$  as follows:

$$(T_1 f, a) := \sum_{n=1}^\infty \left[ \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right] a_n. \quad (3.14)$$

Then we can rewrite (3.1) and (3.2) as follows:

$$(T_1 f, a) < k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \quad (3.15)$$

$$\|T_1 f\|_{p, \Psi_\beta^{1-p}} < k(\sigma) \|f\|_{p, \Phi_\delta}. \quad (3.16)$$

Define the norm of operator  $T_1$  as follows:

$$\|T_1\| := \sup_{f(\neq 0) \in L_{p, \Phi_\delta}(\mathbf{R}_+)} \frac{\|T_1 f\|_{p, \Psi_\beta^{1-p}}}{\|f\|_{p, \Phi_\delta}}.$$

Then by (3.16), it follows that  $\|T_1\| \leq k(\sigma)$ . Since by Theorem 3.2, the constant factor in (3.16) is the best possible, we have

$$\|T_1\| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta\left(\frac{\sigma}{\gamma}, \frac{\alpha + \rho}{2\rho}\right). \quad (3.17)$$

Assuming that  $a = \{a_n\}_{n=1}^\infty \in l_{q, \Psi_\beta}$  and setting

$$h(x) := \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n, \quad x \in \mathbf{R}_+,$$

we can rewrite (3.3) as  $\|h\|_{q, \Phi_\delta^{1-q}} < k(\sigma) \|a\|_{q, \Psi_\beta} < \infty$ , namely,  $h \in L_{q, \Phi_\delta^{1-q}}(\mathbf{R}_+)$ .

**Definition 3.4.** Define a half-discrete Hardy-Hilbert-type operator  $T_2 : l_{q, \Psi_\beta} \rightarrow L_{q, \Phi_\delta^{1-q}}(\mathbf{R}_+)$  as follows: For any  $a = \{a_n\}_{n=1}^\infty \in l_{q, \Psi_\beta}$ , there exists a unique representation

$$T_2 a = h \in L_{q, \Phi_\delta^{1-q}}(\mathbf{R}_+).$$

Define the formal inner product of  $T_2 a$  and  $f \in L_{p, \Phi_\delta}(\mathbf{R}_+)$  as follows:

$$(T_2 a, f) := \int_0^\infty \left[ \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right] f(x) dx. \quad (3.18)$$



Then we can rewrite (3.1) and (3.3) as follows:

$$(T_2 a, f) < k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \quad (3.19)$$

$$\|T_2 a\|_{q, \Phi_\delta^{1-q}} < k(\sigma) \|a\|_{q, \Psi_\beta}. \quad (3.20)$$

Define the norm of operator  $T_2$  as follows:

$$\|T_2\| := \sup_{a(\neq 0) \in l_{q, \Psi}} \frac{\|T_2 a\|_{q, \Phi_\delta^{1-q}}}{\|a\|_{q, \Psi_\beta}}.$$

Then by (3.20), we find  $\|T_2\| \leq k(\sigma)$ . Since by Theorem 3.2, the constant factor in (3.20) is the best possible, we obtain

$$\|T_2\| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}) = \|T_1\|. \quad (3.21)$$

## 4 Some Equivalent Reverse Inequalities

In the following, we also set

$$\tilde{\Phi}_\delta(x) := (1 - \theta_\delta(\sigma, x)) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} \quad (x \in \mathbf{R}_+).$$

For  $0 < p < 1$  or  $p < 0$ , we still use the formal symbols  $\|f\|_{p, \Phi_\delta}$ ,  $\|f\|_{p, \tilde{\Phi}_\delta}$  and  $\|a\|_{q, \Psi_\beta}$  et al.

**Theorem 4.1.** *If  $\rho > \max\{0, -\alpha\}$ ,  $0 < \gamma < \sigma \leq 1$ ,  $k(\sigma)$  is indicated by (2.1), and  $U(\infty) = V(\infty) = \infty$ , then for  $p < 0$ ,  $0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $k(\sigma)$ :*

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \quad (4.1)$$

$$J_1 = \sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p, \Phi_\delta}, \quad (4.2)$$

$$J_2 = \left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) \|a\|_{q, \Psi_\beta}. \quad (4.3)$$

*Proof.* By the reverse weighted Hölder inequality (cf. [48]), since  $p < 0$ , similarly to the way we obtained (3.4) and (3.5), we have

$$\begin{aligned} & \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p \\ & \leq \frac{(\varpi_\delta(\sigma, n))^{p-1}}{(V_n - \beta)^{p\sigma-1} v_{n+1}} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_{n+1}}{(V_n - \beta)^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx. \end{aligned}$$

Then by (2.9) and the Lebesgue term by term integration theorem, it follows that

$$\begin{aligned} J_1 &\geq (k(\sigma))^{\frac{1}{q}} \left[ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x) \mathbf{v}_{n+1}}{(V_n - \beta)^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}} \\ &= (k(\sigma))^{\frac{1}{q}} \left[ \int_0^{\infty} \omega_{\delta}(\sigma, x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Then by (2.7), we have (4.2).

By the reverse Hölder inequality (cf. [48]), we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \left[ \frac{\mathbf{v}_{n+1}^{\frac{1}{p}}}{(V_n - \beta)^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} f(x) dx \right] \left[ \frac{(V_n - \beta)^{\frac{1}{p}-\sigma} a_n}{\mathbf{v}_{n+1}^{\frac{1}{p}}} \right] \\ &\geq J_1 \|a\|_{q, \Psi_{\beta}}. \end{aligned} \quad (4.4)$$

Then by (4.2), we derive (4.1). On the other hand, assuming that (4.1) is valid, we set  $a_n$  as in Theorem 3.1. Then we obtain

$$J_1^p = \|a\|_{q, \Psi_{\beta}}^q.$$

If  $J_1 = \infty$ , then (4.2) is trivially valid. If  $J_1 = 0$ , then (4.2) is still not valid. Suppose that  $0 < J_1 < \infty$ . By (4.1), it follows that

$$\begin{aligned} \|a\|_{q, \Psi_{\beta}}^q &= J_1^p = I > k(\sigma) \|f\|_{p, \Phi_{\delta}} \|a\|_{q, \Psi_{\beta}}, \\ \|a\|_{q, \Psi_{\beta}}^{q-1} &= J_1 > k(\sigma) \|f\|_{p, \Phi_{\delta}}, \end{aligned}$$

and then (4.2) follows, which is equivalent to (4.1).

Applying again the weighted reverse Hölder inequality (cf. [48]), since  $0 < q < 1$ , similarly to how we obtained (3.7) and (3.8), we have

$$\begin{aligned} &\left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} a_n \right]^q \\ &\geq \frac{(\omega_{\delta}(\sigma, x))^{q-1}}{U^{q\delta\sigma-1}(x) \mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} \frac{(V_n - \beta)^{(1-\sigma)(q-1)} \mu(x)}{U^{1-\delta\sigma}(x) \mathbf{v}_{n+1}^{q-1}} a_n^q. \end{aligned}$$

Then, by (2.7) and the Lebesgue term by term integration theorem, it follows that

$$\begin{aligned} J_2 &> (k(\sigma))^{\frac{1}{p}} \left[ \int_0^{\infty} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} \frac{(V_n - \beta)^{(1-\sigma)(q-1)} \mu(x)}{U^{1-\delta\sigma}(x) \mathbf{v}_{n+1}^{q-1}} a_n^q dx \right]^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} \bar{\omega}_{\delta}(\sigma, n) \frac{(V_n - \beta)^{q(1-\sigma)-1}}{\mathbf{v}_{n+1}^{q-1}} a_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

Hence, by (2.9), we have (4.3).

By the reverse Hölder inequality (cf. [48]), we get

$$\begin{aligned} I &= \int_0^\infty \left( \frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x) \right) \left[ \frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right] dx \\ &\geq \|f\|_{p, \Phi_\delta} J_2. \end{aligned} \quad (4.5)$$

Thus by (4.3), we obtain (4.1). On the other hand, assuming that (4.3) is valid, we set  $f(x)$  as in Theorem 4.1. Then we derive that

$$J_2^q = \|f\|_{p, \Phi_\delta}^p.$$

If  $J_2 = \infty$ , then (4.3) is trivially valid. If  $J_2 = 0$ , then (4.3) remains impossible. Suppose that  $0 < J_2 < \infty$ . By (4.1), it follows that

$$\begin{aligned} \|f\|_{p, \Phi_\delta}^p &= J_2^q = I > k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \\ \|f\|_{p, \Phi_\delta}^{p-1} &= J_2 > k(\sigma) \|a\|_{q, \Psi_\beta}, \end{aligned}$$

and then (4.3) follows, which is equivalent to (4.1).

Therefore, inequalities (4.1), (4.2) and (4.3) are equivalent.

For  $\varepsilon \in (0, \frac{q(\sigma-\gamma)}{2})$ , we set  $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q}$ , and  $\tilde{f} = \tilde{f}(x)$ ,  $x \in \mathbf{R}_+$ ,  $\tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$ ,

$$\begin{aligned} \tilde{f}(x) &= \begin{cases} U^{\delta(\tilde{\sigma}+\varepsilon)-1}(x)\mu(x), & 0 < x^\delta \leq 1 \\ 0, & x^\delta > 0 \end{cases}, \\ \tilde{a}_n &= (V_n - \beta)^{\tilde{\sigma}-1} v_{n+1} = (V_n - \beta)^{\sigma-\frac{\varepsilon}{q}-1} v_{n+1}, \quad n \in \mathbf{N}. \end{aligned}$$

By (2.11), (3.12) and (2.7), we obtain

$$\begin{aligned} \|\tilde{f}\|_{p, \Phi_\delta} \|\tilde{a}\|_{q, \Psi_\beta} &= \frac{1}{\varepsilon} U^{\frac{\delta\varepsilon}{p}}(1) \left[ \frac{1}{(v_1 - \beta)^\varepsilon} + \varepsilon O(1) \right]^{\frac{1}{q}}, \\ \tilde{I} &= \sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \tilde{a}_n \tilde{f}(x) dx \\ &= \int_{\{x>0; 0<x^\delta \leq 1\}} \omega_\delta(\tilde{\sigma}, x) \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx \\ &\leq k(\tilde{\sigma}) \int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx \\ &= \frac{1}{\varepsilon} k(\sigma - \frac{\varepsilon}{q}) U^{\delta\varepsilon}(1). \end{aligned}$$

If there exists a positive constant  $K \geq k(\sigma)$ , such that (4.1) is valid when replacing  $k(\sigma)$  to  $K$ , then in particular, we have

$$\varepsilon \tilde{I} > \varepsilon K \|\tilde{f}\|_{p, \Phi_\delta} \|\tilde{a}\|_{q, \Psi_\beta},$$

namely,

$$k(\sigma - \frac{\varepsilon}{q})U^{\delta\varepsilon}(1) > K \cdot U^{\frac{\delta\varepsilon}{p}}(1) \left[ \frac{1}{(v_1 - \beta)^\varepsilon} + \varepsilon O(1) \right]^{\frac{1}{q}}.$$

It follows that  $k(\sigma) \geq K(\varepsilon \rightarrow 0^+)$ . Hence,  $K = k(\sigma)$  is the best possible constant factor of (4.1).

The constant factor  $k(\sigma)$  in (4.2) ((4.3)) is still the best possible. Otherwise, we would reach a contradiction by (4.4) ((4.5)) that the constant factor in (4.1) is not the best possible.  $\square$

**Theorem 4.2.** *With the assumptions of Theorem 4.1, if*

$$0 < p < 1, \quad 0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi_\beta} < \infty,$$

*then we have the following equivalent inequalities with the best possible constant factor  $k(\sigma)$ :*

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p, \tilde{\Phi}_\delta} \|a\|_{q, \Psi_\beta}, \quad (4.6)$$

$$J_1 = \sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p, \tilde{\Phi}_\delta}^p, \quad (4.7)$$

$$\begin{aligned} J &:= \left\{ \int_0^{\infty} \frac{(1 - \theta_\delta(\sigma, x))^{1-q} \mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} \\ &> k(\sigma) \|a\|_{q, \Psi_\beta}. \end{aligned} \quad (4.8)$$

*Proof.* By the reverse weighted Hölder inequality (cf. [48]), since  $0 < p < 1$ , similarly to as we obtained (3.4) and (3.5), we have

$$\begin{aligned} &\left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p \\ &\geq \frac{(\omega_\delta(\sigma, n))^{p-1}}{(V_n - \beta)^{p\sigma-1} v_{n+1}} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_{n+1}}{(V_n - \beta)^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx. \end{aligned}$$

In view of (2.9) and the Lebesgue term by term integration theorem, we find

$$\begin{aligned} J_1 &\geq (k(\sigma))^{\frac{1}{q}} \left[ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_{n+1}}{(V_n - \beta)^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}} \\ &= (k(\sigma))^{\frac{1}{q}} \left[ \int_0^{\infty} \omega_\delta(\sigma, x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Then by (2.10), we have (4.7).

By the reverse Hölder inequality (cf. [48]), we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \left[ \frac{v_{n+1}^{\frac{1}{p}}}{(V_n - \beta)^{\frac{1}{p} - \sigma}} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} f(x) dx \right] \left[ \frac{(V_n - \beta)^{\frac{1}{p} - \sigma} a_n}{v_{n+1}^{\frac{1}{p}}} \right] \\ &\geq J_1 \|a\|_{q, \Psi_{\beta}}. \end{aligned} \quad (4.9)$$

Then by (4.7), we have (4.6). On the other hand, assuming that (4.6) is valid, we set  $a_n$  as in Theorem 3.1. Then we find  $J_1^p = \|a\|_{q, \Psi_{\beta}}^q$ . If  $J_1 = \infty$ , then (4.7) is trivially valid; if  $J_1 = 0$ , then (4.7) keeps impossible. Suppose that  $0 < J_1 < \infty$ . By (4.6), it follows that

$$\begin{aligned} \|a\|_{q, \Psi}^q &= J_1^p = I > k(\sigma) \|f\|_{p, \tilde{\Phi}_{\delta}} \|a\|_{q, \Psi_{\beta}}, \\ \|a\|_{q, \Psi}^{q-1} &= J_1 > k(\sigma) \|f\|_{p, \tilde{\Phi}_{\delta}}, \end{aligned}$$

and then (4.7) follows, which is equivalent to (4.6).

Similarly, by the reverse weighted Hölder inequality (cf. [48]), since  $q < 0$ , we have

$$\begin{aligned} &\left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} a_n \right]^q \\ &\leq \frac{(\omega_{\delta}(\sigma, x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} \frac{(V_n - \beta)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)v_{n+1}^{q-1}} a_n^q. \end{aligned}$$

Therefore, by (2.10) and the Lebesgue term by term integration theorem, it follows that

$$\begin{aligned} J &> (k(\sigma))^{\frac{1}{p}} \left[ \int_0^{\infty} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} \frac{(V_n - \beta)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)v_{n+1}^{q-1}} a_n^q dx \right]^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} \varpi_{\delta}(\sigma, n) \frac{(V_n - \beta)^{q(1-\sigma)-1}}{v_{n+1}^{q-1}} a_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

Hence, by (2.9), we have (4.8).

By the reverse Hölder inequality (cf. [48]), we have

$$\begin{aligned} I &= \int_0^{\infty} \left[ (1 - \theta_{\delta}(\sigma, x))^{\frac{1}{p}} \frac{U^{\frac{1}{q} - \delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x) \right] \\ &\times \left[ \frac{(1 - \theta_{\delta}(\sigma, x))^{\frac{-1}{p}} \mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q} - \delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^{\gamma}}} a_n \right] dx \geq \|f\|_{p, \tilde{\Phi}_{\delta}} J. \end{aligned} \quad (4.10)$$

Then by (4.8), we have (4.6). On the other hand, assuming that (4.6) is valid, we set  $f(x)$  as in Theorem 3.1. Then we derive that  $J^q = \|f\|_{p, \tilde{\Phi}_{\delta}}^p$ . If  $J = \infty$ , then (4.8) is trivially valid; if  $J = 0$ , then (4.8) is still not valid. Suppose that  $0 < J < \infty$ . By (4.6), it follows that

$$\begin{aligned} \|f\|_{p, \tilde{\Phi}_{\delta}}^p &= J^q = I > k(\sigma) \|f\|_{p, \tilde{\Phi}_{\delta}} \|a\|_{q, \Psi_{\beta}}, \\ \|f\|_{p, \tilde{\Phi}_{\delta}}^{p-1} &= J > k(\sigma) \|a\|_{q, \Psi_{\beta}}, \end{aligned}$$

and then (4.8) follows, which is equivalent to (4.6).

Therefore, inequalities (4.6), (4.7) and (4.8) are equivalent.

For  $\varepsilon \in (0, \frac{p(\sigma-\gamma)}{2})$ , we set  $\tilde{\sigma} = \sigma + \frac{\varepsilon}{p}$ , and  $\tilde{f} = \tilde{f}(x), x \in \mathbf{R}_+, \tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$ ,

$$\begin{aligned}\tilde{f}(x) &= \begin{cases} U^{\delta\tilde{\sigma}-1}(x)\mu(x), & 0 < x^\delta \leq 1 \\ 0, & x^\delta > 1 \end{cases}, \\ \tilde{a}_n &= (V_n - \beta)^{\tilde{\sigma}-\varepsilon-1}v_{n+1} = (V_n - \beta)^{\sigma-\frac{\varepsilon}{q}-1}v_{n+1}, n \in \mathbf{N}.\end{aligned}$$

By (2.10), (2.11) and (3.12), we obtain

$$\begin{aligned}& ||\tilde{f}||_{p, \tilde{\Phi}_\delta} ||\tilde{a}||_{q, \Psi_\beta} \\&= \left[ \int_{\{x>0; 0<x^\delta \leq 1\}} (1 - O((U(x))^{\frac{\delta}{2}(\sigma-\gamma)})) \frac{\mu(x)dx}{U^{1-\delta\varepsilon}(x)} \right]^{\frac{1}{p}} \left[ \sum_{n=1}^\infty \frac{v_{n+1}}{(V_n - \beta)^{1+\varepsilon}} \right]^{\frac{1}{q}} \\&= \frac{1}{\varepsilon} \left( U^{\delta\varepsilon}(1) - \varepsilon O_1(1) \right)^{\frac{1}{p}} \left[ \frac{1}{(v_1 - \beta)^\varepsilon} + \varepsilon O(1) \right]^{\frac{1}{q}}, \\ \tilde{I} &= \sum_{n=1}^\infty \int_0^\infty \frac{\csc h(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \tilde{a}_n \tilde{f}(x) dx \\&= \sum_{n=1}^\infty \left[ \int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\csc h(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{(V_n - \beta)^{\tilde{\sigma}} \mu(x)}{U^{1-\delta\tilde{\sigma}}(x)} dx \right] \frac{v_{n+1}}{(V_n - \beta)^{1+\varepsilon}} \\&\leq \sum_{n=1}^\infty \left[ \int_0^\infty \frac{\csc h(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} \frac{(V_n - \beta)^{\tilde{\sigma}} \mu(x)}{U^{1-\delta\tilde{\sigma}}(x)} dx \right] \frac{v_{n+1}}{(V_n - \beta)^{1+\varepsilon}} \\&= \sum_{n=1}^\infty \varpi_\delta(\tilde{\sigma}, n) \frac{v_{n+1}}{(V_n - \beta)^{1+\varepsilon}} = k(\tilde{\sigma}) \sum_{n=1}^\infty \frac{v_{n+1}}{(V_n - \beta)^{1+\varepsilon}} \\&= \frac{1}{\varepsilon} k(\sigma + \frac{\varepsilon}{p}) \left[ \frac{1}{(v_1 - \beta)^\varepsilon} + \varepsilon O(1) \right].\end{aligned}$$

If there exists a positive constant  $K \geq k(\sigma)$ , such that (4.1) is valid when replacing  $k(\sigma)$  by  $K$ , then in particular, we have

$$\varepsilon \tilde{I} > \varepsilon K ||\tilde{f}||_{p, \tilde{\Phi}_\delta} ||\tilde{a}||_{q, \Psi_\beta},$$

namely,

$$\begin{aligned}& k(\sigma + \frac{\varepsilon}{p}) \left[ \frac{1}{(v_1 - \beta)^\varepsilon} + \varepsilon O(1) \right] \\&> K \left( U^{\delta\varepsilon}(1) - \varepsilon O_1(1) \right)^{\frac{1}{p}} \left[ \frac{1}{(v_1 - \beta)^\varepsilon} + \varepsilon O(1) \right]^{\frac{1}{q}}.\end{aligned}$$

It follows that  $k(\sigma) \geq K(\varepsilon \rightarrow 0^+)$ . Hence,  $K = k(\sigma)$  is the best possible constant factor of (4.6).

The constant factor  $k(\sigma)$  in (4.7) ((4.8)) is still the best possible. Otherwise, we would reach a contradiction by (4.9) ((4.10)) that the constant factor in (4.6) is not the best possible.  $\square$

## 5 Some Corollaries

For  $\delta = 1$  in Theorem 3.2, Theorem 4.1 and Theorem 4.2, the following inequalities with the non-homogeneous kernel hold true:

**Corollary 5.1.** *If  $\rho > \max\{0, -\alpha\}$ ,  $0 < \gamma < \sigma \leq 1$ ,  $k(\sigma)$  is indicated by (2.2), and  $U(\infty) = V(\infty) = \infty$ , then*

(i) *for  $p > 1$ ,  $0 < \|f\|_{p, \Phi_1}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^\gamma(x)(V_n - \beta)^\gamma)}{e^{\alpha U^\gamma(x)(V_n - \beta)^\gamma}} a_n f(x) dx < k(\sigma) \|f\|_{p, \Phi_1} \|a\|_{q, \Psi_\beta}, \quad (5.1)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^\gamma(x)(V_n - \beta)^\gamma)}{e^{\alpha U^\gamma(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p < k(\sigma) \|f\|_{p, \Phi_1}, \quad (5.2)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^\gamma(x)(V_n - \beta)^\gamma)}{e^{\alpha U^\gamma(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} < k(\sigma) \|a\|_{q, \Psi_\beta}; \quad (5.3)$$

(ii) *for  $p < 0$ ,  $0 < \|f\|_{p, \Phi_1}, \|a\|_{q, \Psi} < \infty$ , we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^\gamma(x)(V_n - \beta)^\gamma)}{e^{\alpha U^\gamma(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p, \Phi_1} \|a\|_{q, \Psi_\beta}, \quad (5.4)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^\gamma(x)(V_n - \beta)^\gamma)}{e^{\alpha U^\gamma(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p, \Phi_1}, \quad (5.5)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^\gamma(x)(V_n - \beta)^\gamma)}{e^{\alpha U^\gamma(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) \|a\|_{q, \Psi_\beta}; \quad (5.6)$$

(iii) *for  $0 < p < 1$ ,  $0 < \|f\|_{p, \Phi_1}, \|a\|_{q, \Psi} < \infty$ , we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^\gamma(x)(V_n - \beta)^\gamma)}{e^{\alpha U^\gamma(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p, \tilde{\Phi}_1} \|a\|_{q, \Psi_\beta}, \quad (5.7)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^\gamma(x)(V_n - \beta)^\gamma)}{e^{\alpha U^\gamma(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p, \tilde{\Phi}_1}, \quad (5.8)$$

$$\begin{aligned} & \left\{ \int_0^{\infty} \frac{(1 - \theta_1(\sigma, x))^{1-q} \mu(x)}{U^{1-q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^\gamma(x)(V_n - \beta)^\gamma)}{e^{\alpha U^\gamma(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} \\ & > k(\sigma) \|a\|_{q, \Psi_\beta}. \end{aligned} \quad (5.9)$$

The above inequalities involve the best possible constant factor  $k(\sigma)$ .

For  $\delta = -1$  in Theorem 3.2, Theorem 4.1 and Theorem 4.2, we have the following inequalities with the homogeneous kernel of degree 0:

**Corollary 5.2.** *If  $\rho > \max\{0, -\alpha\}$ ,  $0 < \gamma < \sigma \leq 1$ ,  $k(\sigma)$  is indicated by (2.2), and  $U(\infty) = V(\infty) = \infty$ , then*

*(i) for  $p > 1$ ,  $0 < \|f\|_{p, \Phi_{-1}}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(\frac{V_n - \beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n - \beta}{U(x)})^\gamma}} a_n f(x) dx < k(\sigma) \|f\|_{p, \Phi_{-1}} \|a\|_{q, \Psi_\beta}, \quad (5.10)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho(\frac{V_n - \beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n - \beta}{U(x)})^\gamma}} f(x) dx \right]^p < k(\sigma) \|f\|_{p, \Phi_{-1}}, \quad (5.11)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1+q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho(\frac{V_n - \beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n - \beta}{U(x)})^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} < k(\sigma) \|a\|_{q, \Psi_\beta}; \quad (5.12)$$

*(ii) for  $p < 0$ ,  $0 < \|f\|_{p, \Phi_{-1}}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(\frac{V_n - \beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n - \beta}{U(x)})^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p, \Phi_{-1}} \|a\|_{q, \Psi_\beta}, \quad (5.13)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho(\frac{V_n - \beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n - \beta}{U(x)})^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p, \Phi_{-1}}, \quad (5.14)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1+q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho(\frac{V_n - \beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n - \beta}{U(x)})^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) \|a\|_{q, \Psi_\beta}; \quad (5.15)$$

*(iii) for  $0 < p < 1$ ,  $0 < \|f\|_{p, \Phi_{-1}}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(\frac{V_n - \beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n - \beta}{U(x)})^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p, \tilde{\Phi}_{-1}} \|a\|_{q, \Psi_\beta}, \quad (5.16)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho(\frac{V_n - \beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n - \beta}{U(x)})^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p, \tilde{\Phi}_{-1}}, \quad (5.17)$$

$$\begin{aligned} & \left\{ \int_0^{\infty} \frac{(1 - \theta_{-1}(\sigma, x))^{1-q} \mu(x)}{U^{1+q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho(\frac{V_n - \beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n - \beta}{U(x)})^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} \\ & > k(\sigma) \|a\|_{q, \Psi_\beta}. \end{aligned} \quad (5.18)$$

The above inequalities involve the best possible constant factor  $k(\sigma)$ .

For  $\alpha = \rho$  in Theorem 3.2, Theorem 4.1 and Theorem 4.2, we have



**Corollary 5.3.** *If  $\rho > 0, 0 < \gamma < \sigma \leq 1$ , and  $U(\infty) = V(\infty) = \infty$ , then*

*(i) for  $p > 1, 0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities with the best possible constant factor*

$$k_1(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} :$$

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n f(x) dx < k_1(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \quad (5.19)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p < k_1(\sigma) \|f\|_{p, \Phi_\delta}, \quad (5.20)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} < k_1(\sigma) \|a\|_{q, \Psi_\beta}; \quad (5.21)$$

*(ii) for  $p < 0, 0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $k_1(\sigma)$ :*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k_1(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \quad (5.22)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k_1(\sigma) \|f\|_{p, \Phi_\delta}, \quad (5.23)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k_1(\sigma) \|a\|_{q, \Psi_\beta}; \quad (5.24)$$

*(iii) for  $0 < p < 1, 0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $k_1(\sigma)$ :*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k_1(\sigma) \|f\|_{p, \tilde{\Phi}_\delta} \|a\|_{q, \Psi_\beta}, \quad (5.25)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k_1(\sigma) \|f\|_{p, \tilde{\Phi}_\delta}, \quad (5.26)$$

$$\begin{aligned} & \left\{ \int_0^{\infty} \frac{(1 - \theta_\delta(\sigma, x))^{1-q} \mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} \\ & > k_1(\sigma) \|a\|_{q, \Psi_\beta}. \end{aligned} \quad (5.27)$$

For  $\gamma = \frac{\sigma}{2}$  in Corollary 5.3, we obtain the following:

**Corollary 5.4.** *If  $\rho > 0, 0 < \sigma \leq 1$ , and  $U(\infty) = V(\infty) = \infty$ , then*

*(i) for  $p > 1, 0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $\frac{\pi^2}{6\sigma\rho^2}$ :*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2}}} a_n f(x) dx < \frac{\pi^2}{6\sigma\rho^2} \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \quad (5.28)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2}}} f(x) dx \right]^p < \frac{\pi^2}{6\sigma\rho^2} \|f\|_{p, \Phi_\delta}, \quad (5.29)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2}}} a_n \right]^q dx \right\}^{\frac{1}{q}} < \frac{\pi^2}{6\sigma\rho^2} \|a\|_{q, \Psi_\beta}; \quad (5.30)$$

*(ii) for  $p < 0, 0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $\frac{\pi^2}{6\sigma\rho^2}$ :*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2}}} a_n f(x) dx > \frac{\pi^2}{6\sigma\rho^2} \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi_\beta}, \quad (5.31)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2}}} f(x) dx \right]^p > \frac{\pi^2}{6\sigma\rho^2} \|f\|_{p, \Phi_\delta}, \quad (5.32)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2}}} a_n \right]^q dx \right\}^{\frac{1}{q}} > \frac{\pi^2}{6\sigma\rho^2} \|a\|_{q, \Psi_\beta}; \quad (5.33)$$

*(iii) for  $0 < p < 1, 0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi_\beta} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $\frac{\pi^2}{6\sigma\rho^2}$ :*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2}}} a_n f(x) dx > \frac{\pi^2}{6\sigma\rho^2} \|f\|_{p, \tilde{\Phi}_\delta} \|a\|_{q, \Psi_\beta}, \quad (5.34)$$

$$\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2}}} f(x) dx \right]^p > \frac{\pi^2}{6\sigma\rho^2} \|f\|_{p, \tilde{\Phi}_\delta}, \quad (5.35)$$

$$\begin{aligned} & \left\{ \int_0^{\infty} \frac{(1 - \theta_\delta(\sigma, x))^{1-q} \mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2}}} a_n \right]^q dx \right\}^{\frac{1}{q}} \\ & > \frac{\pi^2}{6\sigma\rho^2} \|a\|_{q, \Psi_\beta}. \end{aligned} \quad (5.36)$$

*Remark 5.5.*

(i) For  $\beta = 0$  in (3.1), the following inequality holds true:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} a_n f(x) dx < k(\sigma) \|f\|_{p, \Phi_{\delta}} \|a\|_{q, \Psi_0}. \quad (5.37)$$

Hence, (3.1) is a more accurate inequality of (5.37) for  $0 < \beta \leq \frac{\nu_1}{2}$ .

(ii) For  $\mu(x) = \nu_n = 1$  in (5.37), we have the following inequality with the best possible constant factor  $k(\sigma)$ :

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(x^{\delta}n)^{\gamma})}{e^{\alpha(x^{\delta}n)^{\gamma}}} a_n f(x) dx \\ & < k(\sigma) \left[ \int_0^{\infty} x^{p(1-\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \end{aligned} \quad (5.38)$$

In particular, for  $\delta = 1$ , we have the following inequality with the non-homogeneous kernel:

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(xn)^{\gamma})}{e^{\alpha(xn)^{\gamma}}} a_n f(x) dx \\ & < k(\sigma) \left[ \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}; \end{aligned} \quad (5.39)$$

for  $\delta = -1$ , we have the following inequality with the homogeneous kernel of degree 0:

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(\frac{n}{x})^{\gamma})}{e^{\alpha(\frac{n}{x})^{\gamma}}} a_n f(x) dx \\ & < k(\sigma) \left[ \int_0^{\infty} x^{p(1+\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \end{aligned} \quad (5.40)$$

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